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On interpolative Hardy-Rogers type cyclic contractions

Mohamed Edraoui [a,d], Amine El koufi [b,c] and Mohamed Aamri [d]

[a] Laboratory of Analysis, Modeling, and Simulation, Department of Mathematics and Computer Sciences, Ben M'Sik Faculty of Sciences, Hassan II University of Casablanca, P.B 7955 Sidi Othmane, Casablanca, Morocco (edraoui.mohamed@gmail.com)
[b] High School of Technology, Ibn Tofail University, BP 240, Kénitra, Morocco (amine.elkoufi@uit.ac.ma)
[c] Lab of PDE's, Algebra and spectral geometry, Faculty of Sciences, Ibn Tofail University, BP 133, Kénitra, Morocco (amine.elkoufi@uit.ac.ma)
[d] Laboratory of Algebra, Analysis and Applications, Department of Mathematics and Computer Sciences, Ben M'Sik Faculty of Sciences, Hassan II University of Casablanca, P.B 7955 Sidi Othmane, Casablanca, Morocco (aamrimohamed9@yahoo.fr)

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Abstract

Recently, Karapınar introduced a new Hardy-Rogers type contractive mapping using the concept of interpolation and proved a fixed point theorem in complete metric space. This new type of mapping, called "interpolative Hardy-Rogers type contractive mapping" is a generalization of Hardy-Rogers's fixed point theorem. Following this direction of research, in this paper, we will present some fixed point results of Hardy-Rogers-type for cyclic mappings on complete metric spaces. Moreover, an example is given to illustrate the usability of the obtained results.

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1. Introduction

The study of cyclic contractions and best proximity points is important in mathematics and has applications in various fields, including optimization, functional analysis, and fixed point algorithms. These concepts provide a broader framework for understanding fixed point theorems beyond standard contraction mappings. Overall, these topics have contributed to the development of fixed-point theory and have practical implications in mathematical modelling and problem-solving. The first result in this area was reported by Kirk-Srinavasan-Veeramani [24] in 2003. Later, many authors continued investigation and more results were obtained, such as, [25, 7, 1, 22, 26, 5]. Researchers continue to explore and expand upon these ideas.

We first define the cyclic map.

Definition 1.1 ([24])**.** Let (E,d) be a metric space and let X and Y be two nonempty subsets of E.

A mapping $T : X \cup Y \to X \cup Y$ is said to be a cyclic mapping provided that

$$T(X) \subseteq Y,\ T(Y) \subseteq X \tag{1.1}$$

In 2003, Kirk-Srinivasan-Veeraman [24] proved the following fixed point theorem for a cyclic map.

Theorem 1.2 ([24])**.** *Let (E,d) be non-empty closed subsets of a complete metric space and let X and Y be two nonempty subsets of E. Suppose that $T : X \cup Y \to X \cup Y$ is said to be a cyclic contraction and there exists $k \in (0,1)$ such that $d(Tx,Ty) \le kd(x,y)$ for all $x \in X$,and $y \in Y$. Then, T has a unique fixed point in $X \cap Y$.*

Also, Karapinar and Erhan [17] gave the following definitions of different types of cyclic contractions and proved the existence of unique fixed points for maps (i)–(iii) below

Definition 1.3 ([17])**.** Let (E,d) be a metric space and let X and Y be two nonempty subsets of E.

A cyclic map $T : X \cup Y \to X \cup Y$ is said to be a:

(i) Kannan type cyclic contraction if there exists $k \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \le k\left[d(x,Tx) + d(y,Ty)\right], \quad \forall x \in X, \forall y \in Y$$

(ii) Chatterjea type cyclic contraction if there exists $k \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \le k\left[d(y,Tx) + d(x,Ty)\right], \quad \forall x \in X, \forall y \in Y$$

(iii) Reich type cyclic contraction if there exists $k \in (0, \frac{1}{3})$ such that

$$d(Tx,Ty) \le k\left[d(x,y) + d(x,Tx) + d(y,Ty)\right], \quad \forall x \in X, \forall y \in Y$$

If (E,d) is a complete metric space, at least one of (i), (ii) and (iii) holds, then it has a unique fixed point.

On the "interpolative Hardy-Rogers type contractive mapping" and its generalization of Hardy-Rogers' fixed point theorem. It's interesting to note that

Erdal Karapınar [17] introduced this new type of mapping by incorporating the concept of interpolation into the Hardy-Rogers framework. This approach likely allows for the generation of intermediate points between known data points and expands the applicability of the original theorem.

Using interpolation to generalize various contractions is a common practice in mathematical research. By integrating interpolation techniques into contraction mappings, researchers can extend the scope of existing theorems and provide a more flexible framework for analyzing fixed points in metric spaces.

The interpolative method has been employed in other research as well to generalize different types of contractions. This demonstrates the versatility and effectiveness of the interpolation approach in expanding the theory of fixed points and we provide new insights into the existence and uniqueness of solutions.

To delve further into the specific details and implications of Karapinar's [17, 11] work and the generalization of other forms of contractions using the interpolative method, I recommend referring to the cited paper. [16, 17, 11, 12, 20, 13, 18, 14, 15, 19, 2, 23, 8, 27, 28, 7, 5, 6]

In 2018 Karapınar [11] proposed a new Kannan-type contractive mapping using the concept of interpolation and proved a fixed point theorem in metric space. This new type of mapping, called "interpolative Kannan-type contractive mapping" is a generalization of Kannan's fixed point theorem.

Theorem 1.4 ([11]). *Let us recall that an interpolative Kannan contraction on a metric space (E, d) is a self-mapping $T : E \to E$ such that there exist $k \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \leq k\,[d(Tx, x)]^{\alpha}[d(Ty, y)]^{1-\alpha} \tag{1.2}$$

$(x, y) \in E \times E$ *with* $x, y \notin Fix(T)$

Then T has a unique fixed point in E.

This theorem has been generalized in 2023 by Edraoui, El koufi [5] for various types of cyclic contractions in a metric space.

Definition 1.5 ([5]). Let (E, d) be a metric space and let X and Y be nonempty subsets of E. A cyclic map $T : X \cup Y \to X \cup Y$ is said to be an interpolative Kannan Type cyclic contraction if there exists $k \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq k\,[d(Tx, x)]^{\alpha}[d(Ty, y)]^{1-\alpha} \tag{1.3}$$

for all $(x, y) \in X \times Y$ with $x, y \notin Fix(T)$.

Theorem 1.6 ([5]). *Let (E, d) be a complete metric space and let X and Y be nonempty subsets of E and let $T : X \cup Y \to X \cup Y$ be interpolative Kannan type cyclic contraction. Then T has a unique fixed point in $X \cap Y$.*

One another of the most interesting the Banach contraction principle Hardy-Rogers gave generalizations of it [9].

Theorem 1.7. *Let (E, d) be a complete metric space and T be a self-mapping of E satisfying the condition for all $x, y \in E$, $d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(x, Tx) + \delta d(y, Tx) + \lambda d(x, y)$*

where $\alpha, \beta, \gamma, \delta, \lambda$ are non-negative and $\alpha + \beta + \gamma + \delta + \lambda < 1$. Then has a unique fixed point in E.

Recently Karapinar [16] proposed a new Hardy-Rogers type contractive mapping using the concept of interpolation and proved a fixed point theorem in metric space. This new type of mapping, called "interpolative Hardy-Rogers type contractive mapping" is a generalization of Hardy-Rogers 's fixed point theorem. The interpolative method has been used in other research to generalize other forms of contractions as well [16, 17, 11, 12, 13, 18, 14, 21, 15, 19, 2, 5] . This method has been found to be a powerful tool in the study of fixed point theory, as it allows for the construction of new classes of contractive mappings and the discovery of new fixed point theorems.

Theorem 1.8 ([16]). *Let (E, d) be a complete metric space. The self-mapping $T : E \to E$ is called an interpolative Hardy-Rogers type contraction if there exists Type contraction if there exists $k \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$d(Tx, Ty) \leq k [d(x, y)]^{\beta} d(Tx, x)]^{\alpha} [d(Ty, y)]^{\gamma} . \left[\frac{1}{2}(d(Tx, y) + d(Ty, x))\right]^{1-\alpha-\beta-\gamma}$$

for each $(x, y) \in E$ with $x, y \notin Fix(T)$. Then the mapping T has a fixed point in E.

2. Main results

The interpolation method has been used to generalize the definition of Hardy-Rogers-type cyclic contraction by incorporating the notion of interpolation. This leads to a more general definition of a Hardy-Rogers-type cyclic contraction that allows for the construction of new classes of contractive mappings and the discovery of new fixed point theorems. The idea is that, by incorporating interpolation, the definition of a Hardy-Roger-type cyclic contraction can be expanded and new properties can be discovered.

Consequently, with the generalization of the definition of Hardy-Rogers-type cyclic contraction via interpolation, it becomes reasonable to anticipate the establishment of a fixed point theorem for this newly introduced class of mappings.

Definition 2.1. Let (E, d) be a metric space and let X and Y be nonempty subsets of E. A cyclic map $T : X \cup Y \to X \cup Y$ is said to be an interpolative Hardy-Rogers type contraction if there exists Type cyclic contraction if there exists $k \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that

$$d(Tx, Ty) \leq k [d(x, y)]^{\beta} d(Tx, x)]^{\alpha} [d(Ty, y)]^{\gamma} . \left[\frac{1}{2}(d(Tx, y) + d(Ty, x))\right]^{1-\alpha-\beta-\gamma} \tag{2.1}$$

for all $(x, y) \in X \times Y$ with $x, y \notin Fix(T)$.

The fixed point theorem for an interpolative Hardy-Rogers type cyclic contraction can be stated as: In a complete metric space if a mapping satisfies certain conditions such as being an interpolative Hardy-Rogers-type cyclic contraction, then it has a unique fixed point.

This theorem can be proved by using the properties of interpolative Hardy-Rogers-type cyclic contraction mappings and the Banach fixed point theorem. By using interpolation, we can construct a new class of contractive mappings with a unique fixed point.

Theorem 2.2. *Let (E, d) be a complete metric space and let X and Y be nonempty subsets of X and let $T : X \cup Y \to X \cup Y$ be interpolative Hardy-Rogers type cyclic contraction. Then T has a unique fixed point in $X \cap Y$.*

Proof. Fix $x \in X$, from (2.1) it follows that which yields that

$$d\left(T^2x, Tx\right) \leq k\left[d\left(Tx, x\right)\right]^{\beta} d\left(T^2x, Tx\right)]^{\alpha}[d\left(Tx, x\right)]^{\gamma}\left[\tfrac{1}{2}\left(d\left(T^2x, x\right) + d\left(Tx, Tx\right)\right)\right]^{1-\alpha-\beta-\gamma}$$

$$\begin{aligned}\left[d\left(T^2x, Tx\right)\right]^{1-\alpha} &\leq k\left[d\left(Tx, x\right)\right]^{\beta+\gamma}\left[\tfrac{1}{2}d\left(T^2x, x\right)\right]^{1-\alpha-\beta-\gamma} \\ &\leq k\left[d\left(Tx, x\right)\right]^{\beta+\gamma}\left[\tfrac{1}{2}\left(d\left(T^2x, Tx\right) + d\left(Tx, x\right)\right)\right]^{1-\alpha-\beta-\gamma}\end{aligned} \tag{2.2}$$

Suppose that $d(Tx, x) < d\left(T^2x, Tx\right)$.

Thus,

$$\frac{1}{2}\left(d\left(T^2x, Tx\right) + d\left(Tx, x\right)\right) \leq d\left(T^2x, Tx\right)$$

Consequently, the inequality (2.2) yields that

$$\begin{aligned}\left[d\left(T^2x, Tx\right)\right]^{1-\alpha} &\leq k\left[d\left(Tx, x\right)\right]^{\beta+\gamma}\left[d\left(T^2x, Tx\right)\right]^{1-\alpha-\beta-\gamma} \\ \left[d\left(T^2x, Tx\right)\right]^{\beta+\gamma} &\leq k\left[d\left(Tx, x\right)\right]^{\beta+\gamma}\end{aligned}$$

So we can conclude that $d(Tx, x) \geq d\left(T^2x, Tx\right)$ which is a contradiction.

Thus, we have

$$\begin{aligned}d\left(T^2x, Tx\right) &\leq d\left(Tx, x\right) \\ \frac{1}{2}\left(d\left(T^2x, Tx\right) + d\left(Tx, x\right)\right) &\leq d\left(Tx, x\right).\end{aligned}$$

Consequently, the inequality (2.2) yields that

$$\left[d\left(T^2x, Tx\right)\right]^{1-\alpha} \leq k\left[d\left(Tx, x\right)\right]^{1-\alpha}$$

Inductively, using this process for all $n \in \mathbb{N}$, we have

$$d\left(T^{n+1}x, T^nx\right) \leq t^n d\left(Tx, x\right).where t = k^{\frac{1}{1-\alpha}} \text{ and clearly } t \in (0, 1).$$

Consequently,

$$\sum_{n=1}^{\infty} d\left(T^{n+1}x, T^nx\right) \leq (\sum_{n=1}^{\infty} t^n) d\left(Tx, x\right) < \infty$$

Obviously, $\{T^nx\}$ is a Cauchy sequence. Then, there exists a $z \in X \cup Y$ such that $T^n x \to z$

Notice that $\left\{T^{2n}x\right\}$ is a sequence in X and $\left\{T^{2n+1}x\right\}$ is a sequence in Y having the same limit z. As X and Y are closed, we conclude $z \in X \cap Y$, that is, $X \cap Y$ is nonempty.

We claim that $Tz = z$. Observe that

$d(z, Tz) = \lim_{n\to\infty} d\left(Tz, T^{2n}x\right) = \lim_{n\to\infty} d\left(Tz, TT^{2n-1}x\right)$

$\leq \lim_{n\to\infty} k\left[d\left(z, T^{2n-1}x\right)\right]^{\beta} d\left(T^{2n}x, T^{2n-1}x\right)]^{\alpha}[d(Tz, z)]^{\gamma}\left[\frac{1}{2}\left(d(Tz, x) + d\left(T^{2n}x, z\right)\right)\right]^{1-\alpha-\beta-\gamma}$

Taking $n \to \infty$ in the inequality above, we derive that $d(z, Tz) = 0$ that is $Tz = z$

To prove the uniqueness of the fixed point z, assume that there exists $w \in X \cup Y$ such that $z \neq w$ and $Tw = w$. Taking into account that T is a cyclic, we get $w \in X \cap Y$

we have

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \leq k[d(z, w)]^{\beta} d(Tz, z)]^{\alpha}[d(Tw, w)]^{\gamma}.\left[\frac{1}{2}(d(Tz, w) + d(Tw, z))\right]^{1-\alpha-\beta-\gamma} \\ &\leq k[d(z, w)]^{\beta} d(z, z)]^{\alpha}[d(w, w)]^{\gamma}.\left[\frac{1}{2}(d(Tz, w) + d(Tw, z))\right]^{1-\alpha-\beta-\gamma} = 0 \end{aligned}$$

which yields that $d(z, w) = 0$. We conclude that $z = w$ and hence z is the unique fixed point of T. □

Example 2.3. Let $E = [0, 2]$ and $T : X \cup Y \to X \cup Y$ defined by

$$Tx = \begin{cases} \frac{7}{8} \text{ if } x \in [0, 1] \\ \frac{x}{3} \text{ if } x \in (1, 2] \end{cases}$$

Suppose that $X = Y = [0, 2]$. Define the function $d : E \times E \to [0, +\infty)$ be the usual metric on $\mathbb{R}$.

We choose $k = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\alpha = \frac{1}{3}$ and $\gamma = \frac{1}{7}$.Then, we have to check that holds. We have to examine the following cases:

Case 1: For $x, y \in [0, 1]$ we obtain $d(Tx, Ty) = 0$, so (2.1) holds.

Case 2: For $x = 0$ and $y = 2$

$d(T0, T2) = \left|\frac{7}{8} - \frac{2}{3}\right| = 0.208333333$

$$\begin{aligned} \frac{1}{2}[d(0, 2)]^{\frac{1}{2}}[d(T0, 0)]^{\frac{1}{7}}[d(T2, 2)]^{\frac{1}{3}}.\left[\frac{1}{2}(d(T0, 2) + d(T2, 0))\right]^{\frac{1}{42}} &= \frac{1}{2}[2]^{\frac{1}{2}}[\frac{7}{8}]^{\frac{1}{7}}[\frac{4}{3}]^{\frac{1}{3}}.\left[\frac{43}{48}\right]^{\frac{1}{42}} \\ &= 0.7615689633 \end{aligned}$$

Thus, (2.1)) holds

From all the above two cases, we obtain that T is interpolative Hardy-Rogers type cyclic contraction. Thus, the assumptions of theorem 2.2 are satisfied, also the mapping T has a fixed point, that is $x = \frac{7}{8}$

Corollary 2.4. *Let X and Y be two non-empty closed subsets of a complete metric space (E, d). Let $T_1 : X \to Y$ and $T_2 : Y \to X$ be two functions.*

Assume that there exists $k \in [0,1)$ *and* $\alpha, \beta, \gamma \in (0,1)$ *with* $\alpha + \beta + \gamma < 1$, *such that*

$$d\left(T_1x, T_2y\right) \leq k\left[d\left(x,y\right)\right]^{\beta} d\left(T_1x, x\right)]^{\alpha}[d\left(T_2y, y\right)]^{\gamma} \cdot \left[\frac{1}{2}\left(d\left(T_1x, y\right) + d\left(T_2y, x\right)\right)\right]^{1-\alpha-\beta-\gamma} \tag{2.2}$$

for all $(x,y) \in X \times Y$ *with* $x, y \notin Fix\left(T_i\right)$ $i = 1,2$. *Then there exists a unique* $z \in X \cap Y$ *such that* $T_1(z) = T_2(z) = z$.

Proof. Let $T : X \cup Y \to X \cup Y$ defined by

$$\begin{cases} T_1(x) & \text{if } x \in X \\ T_2(x) & \text{if } x \in Y \end{cases}$$

Then T be interpolative Hardy-Rogers type cyclic contraction on complete metric space (E,d), we can now apply Theorem 2.2 to deduce that T has a fixed point $z \in X \cap Y$ such that $T_1(z) = T_2(z) = z$. □

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