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MEAN ERGODIC COMPOSITION OPERATORS ON GENERALIZED FOCK SPACES

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Abstract. Every bounded composition operator C_ψ defined by an analytic symbol ψ on the complex plane when acting on generalized Fock spaces $\mathcal{F}_\varphi^p, 1 \leq p \leq \infty$, is power bounded. Mean ergodic and uniformly mean ergodic composition operators on the spaces are characterized. The set of periodic points of these operators is also determined.

1. Introduction and preliminaries

Given an entire function ψ on the complex plane $\mathbb{C}$, the composition operator induced by ψ is defined by $C_\psi f = f \circ \psi$ for each entire function f. The study of composition operators acting between spaces of analytic functions defined on the disc or on the complex plane has quite a long and rich history. We refer the reader to the monographs [10] and [20]. Many properties of composition operators on spaces of entire functions have been also investigated. For instance, in the frame of Fock spaces, in 2003, Carswell, MacCluer and Schuster [8] characterized bounded and compact composition operators on the classical Fock spaces $\mathcal{F}^p$, $0 < p < \infty$. They showed that only the class of linear mappings $\psi(z) = az + b, |a| \leq 1$ and $b = 0$ whenever $|a| = 1$ induces bounded composition operators. Compactness of C_ψ was described by the strict requirement $|a| < 1$. In 2008, Guo and Izuchi [12] studied various aspects of the operators on Fock type spaces, in this case the symbols which induce the bounded composition operators are the same. Very recently, the first two authors [19, 18] studied the operator on generalized Fock spaces $\mathcal{F}_\varphi^p$ where the weight function φ is smooth, radial and it grows faster than the Gaussian weight function $\frac{|z|^2}{2}$. They described various topological and dynamical properties of the operator on $\mathcal{F}_\varphi^p$. The precise definition of the space $\mathcal{F}_\varphi^p$ and the conditions on the weight function will be given below. We state now their result about continuity and compactness in the setting which is relevant for our discussions below.

Theorem 1.1. *(Theorem 2.1 of* [18], *Theorem 2.1 of* [19]*)*
Let $1 \leq p \leq \infty$ and ψ be a nonconstant analytic map on the complex plane $\mathbb{C}$. Then the operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$ is

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i) *bounded if and only if* $\psi(z) = az + b$ *for some complex numbers* a *and* b *such that* $|a| \leq 1$*, and* $b = 0$ *whenever* $|a| = 1$.
ii) *compact if and only if* $\psi(z) = az + b$ *for some complex numbers* a *and* b *such that* $|a| < 1$.

We use below the notation $U(z) \simeq V(z)$ meaning that there is a constant $C > 0$ such that $(1/C)V(z) \leq U(z) \leq CV(z)$ for all z in the set in which the functions are defined.

The generalized Fock spaces are defined as follows. Let $\varphi : [0, \infty) \to [0, \infty)$ be a twice continuously differentiable function. We extend φ to the whole complex plane by setting $\varphi(z) = \varphi(|z|)$. We further assume that its Laplacian satisfies $\Delta\varphi > 0$, and we set

$$\tau(z) \simeq \begin{cases} 1, & |z| \leq 1 \\ (\Delta\varphi(z))^{-1/2}, & |z| > 1 \end{cases}$$

where τ is a radial differentiable function satisfying the admissibility conditions

$$\lim_{r\to\infty} \tau(r) = 0 \quad \text{and} \quad \lim_{r\to\infty} \tau'(r) = 0,$$

and there exists a constant $C > 0$ such that $\tau(r)r^C$ increases for large r or

$$\lim_{r\to\infty} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

There are many concrete examples of weight functions φ that satisfy the above smoothness and admissibility conditions. The power functions $\varphi_\alpha(r) = r^\alpha,\ \alpha \geq 2$, the exponential functions such as $\varphi_\beta(r) = e^{\beta r},\ \ \beta > 0$, and the supper exponential functions $\varphi(r) = e^{e^r}$ are all typical examples of such weight functions.

The generalized Fock spaces $\mathcal{F}^p_\varphi, 1 \leq p \leq \infty$, associated with φ are the spaces consisting of all entire functions f such that

$$\|f\|_{(\varphi,p)} = \begin{cases} \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} dA(z) \right)^{1/p}, & 1 \leq p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-\varphi(z)}, & p = \infty \end{cases}$$

and dA denotes the usual Lebesgue area measure on $\mathbb{C}$. These spaces have been studied in various contexts; see for instance [4, 6, 9, 17].

For $1 \leq p < \infty$, we can write the above definition in polar coordinates as follows.

$$\|f\|^p_{(\varphi,p)} = \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} dA(z) = 2\pi \int_0^\infty \left[\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} \right] re^{-p\varphi(r)} dr.$$

For $0 < r < \infty$, if we write $M_p(f, r) = \left[\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} \right]^{\frac{1}{p}}$, then

$$\|f\|^p_{(\varphi,p)} = 2\pi \int_0^\infty M_p^p(f, r) re^{-p\varphi(r)} dr.$$

We define $M_\infty(f, r) := \sup_{|z|=r} |f(z)|$. For each $1 \leq p \leq \infty$ the function $M_p(f, r)$ is increasing for $r \in [0, \infty)$.

We also consider the subspace $\mathcal{F}_\varphi^0$ of $\mathcal{F}_\varphi^\infty$ defined by

$$\mathcal{F}_\varphi^0 = \{f \in \mathcal{F}_\varphi^\infty : \lim_{|z|\to\infty} |f(z)|e^{-\varphi(z)} = 0\}.$$

This subspace is closed in $\mathcal{F}_\varphi^\infty$ and it contains polynomials. Moreover, the polynomials are dense in $\mathcal{F}_\varphi^0$, and $\mathcal{F}_\varphi^\infty$ is canonically isomorphic to the bidual of $\mathcal{F}_\varphi^0$. These statements can be seen in a more general context in [3].

We recall some definitions. Let X be a Banach space, and T a bounded operator acting on X. We denote the n-th ergodic mean T_n by

$$T_n := \frac{1}{n}\sum_{m=1}^{n} T^m.$$

T is said to be *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. We say that T is *mean ergodic* if there exists a bounded operator P on X such that for each f in X

$$\lim_{n\to\infty} \|T_n f - Pf\| = 0$$

and *uniformly mean ergodic* if the pointwise convergence above is uniform. That is

$$\lim_{n\to\infty} \|T_n - P\| = 0.$$

The standard references for mean ergodic operators are the books of Krengel [13] and Yosida [22]; see also [1]. Mean ergodicity of composition operator was studied in different spaces; Bonet and Domański [5] on spaces of analytic functions over some domain in a Stain manifold, E. Wolf [21] on $H_v^\infty(\mathbb{D})$, Beltrán-Meneu, Gómez-Collado, Jordá and Jornet [2] on a disc algebra $A(\mathbb{D})$ and the space $H^\infty\mathbb{D})$ of bounded analytic function. Mean ergodic multiplication operators on $H_v^\infty(\mathbb{D})$ were also investigated by Bonet and Ricker in [7].

We now state the main result of our article. Its proof will be presented in the next section.

Theorem 1.2. *Let $1 \le p \le \infty$ or $p = 0$ and C_ψ be a bounded composition operator on the generalized Fock space $\mathcal{F}_\varphi^p$.*

(1) *C_ψ is power bounded.*
(2) *If $\psi(z) = az + b$, $|a| < 1$, then C_ψ is uniformly mean ergodic.*
(3) *If $\psi(z) = az$, $|a| = 1$ and a is a root of the unity, then C_ψ is uniformly mean ergodic and every $f \in \mathcal{F}_\varphi^p$ is a periodic point of C_ψ.*
(4) *If $\psi(z) = az$, $|a| = 1$ and a is not a root of the unity, then C_ψ is mean ergodic but not uniformly mean ergodic if $1 \le p < \infty$ or $p = 0$, but it is not mean ergodic on $\mathcal{F}_\varphi^\infty$.*
(5) *The constant functions are the only periodic points for C_ψ in $\mathcal{F}_\varphi^p$ if $\psi(z) = az + b, |a| < 1$ or $\psi(z) = az, |a| = 1$ and a is not a root of the unity.*

Statement (1) is proved in Theorem 2.1. Statement (2) is Theorem 2.5 and statement (3) is Proposition 2.6. Result (4) follows from Theorems 2.7 and 2.8. Finally, the last statement (5) is a consequence of Proposition 2.10.

We remark that as easily seen from the proofs in the next section, Theorem 1.2 is also valid on the classical Fock spaces $\mathcal{F}^p,\ 1 \le p \le \infty$.

2. Proof of the main result

The proof of the main theorem, Theorem 1.2, will be presented in several statements.

Theorem 2.1. *Let $1 \le p \le \infty$ and C_ψ be bounded on $\mathcal{F}^p_\varphi$. Then C_ψ is power bounded.*

Proof. We split the proof into two cases.
Case 1: $1 \le p < \infty$.
We first consider the case when $|a| < 1$. In polar coordinates $a = |a|e^{i\phi}$ for some ϕ. For each $n \in \mathbb{N}$ consider a translation $\tau_n(z) = z + \frac{b(1-a^n)}{1-a}$. Then for any $f \in \mathcal{F}^p_\varphi$ and $z = re^{it} \in \mathbb{C}$,

$$\int_0^{2\pi} \left| f\left(a^n re^{it} + \frac{b(1-a^n)}{1-a} \right) \right|^p \frac{dt}{2\pi} = \int_0^{2\pi} \left| f \circ \tau_n(a^n re^{it}) \right|^p \frac{dt}{2\pi}$$
$$= \int_0^{2\pi} \left| f \circ \tau_n((|a|^n r)e^{i(t+n\phi)}) \right|^p \frac{dt}{2\pi} = M_p^p(f \circ \tau_n, (|a|^n r)).$$

Since M_p^p is increasing and $|a|^n \le |a|$, we have

$$M_p^p(f \circ \tau_n, (|a|^n r)) \le M_p^p(f \circ \tau_n, (|a| r)) = \int_0^{2\pi} \left| f \circ \tau_n(|a| re^{i(t+n\phi)}) \right|^p \frac{dt}{2\pi}$$
$$= \int_0^{2\pi} |f \circ \tau_n((|a|e^{i\phi})re^{i(t+(n-1)\phi)})|^p \frac{dt}{2\pi} = \int_{(n-1)\phi}^{2\pi+(n-1)\phi} |f \circ \tau_n((\underbrace{|a|e^{i\phi}}_{=a})re^{it})|^p \frac{dt}{2\pi}$$
$$= \int_{(n-1)\phi}^{2\pi+(n-1)\phi} \left| f\left(are^{it} + \frac{b(1-a^n)}{1-a} \right) \right|^p \frac{dt}{2\pi} = \int_0^{2\pi} \left| f\left(are^{it} + \frac{b(1-a^n)}{1-a} \right) \right|^p \frac{dt}{2\pi}.$$

Thus

$$\int_0^{2\pi} \left| f\left(a^n re^{it} + \frac{b(1-a^n)}{1-a} \right) \right|^p \frac{dt}{2\pi} \le \int_0^{2\pi} \left| f\left(are^{it} + \frac{b(1-a^n)}{1-a} \right) \right|^p \frac{dt}{2\pi}.$$

Now, for any $n \in \mathbb{N}$ and $f \in \mathcal{F}^p_\varphi$,

$$\|C_\psi^n f\|^p_{(\varphi,p)} = \int_{\mathbb{C}} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a} \right) \right|^p e^{-p\varphi(z)} dA(z)$$
$$= 2\pi \int_0^\infty \int_0^{2\pi} \left| f\left(a^n re^{it} + \frac{b(1-a^n)}{1-a} \right) \right|^p \frac{dt}{2\pi} re^{-p\varphi(r)} dr$$
$$\le 2\pi \int_0^\infty \int_0^{2\pi} \left| f\left(are^{it} + \frac{b(1-a^n)}{1-a} \right) \right|^p \frac{dt}{2\pi} re^{-p\varphi(r)} dr$$
$$= \int_{\mathbb{C}} \left| f\left(az + \frac{b(1-a^n)}{1-a} \right) \right|^p e^{-p\varphi(z)} dA(z). \tag{2.1}$$

From (2.1) and the fact that $|a|<1$,

$$\begin{aligned}
\|C_\psi^n f\|^p_{(\varphi,p)} &\le \int_{\mathbb{C}} \left|f\left(az+\frac{b(1-a^n)}{1-a}\right)\right|^p e^{-p\varphi(z)}dA(z)\\
&= \int_{\mathbb{C}} \left|f\left(az+\frac{b(1-a^n)}{1-a}\right)\right|^p e^{-p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)} e^{p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)-p\varphi(z)}dA(z)\\
&\le \sup_{z\in\mathbb{C}} e^{p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)-p\varphi(z)} \int_{\mathbb{C}} \left|f\left(az+\frac{b(1-a^n)}{1-a}\right)\right|^p e^{-p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)}dA(z)\\
&= \frac{1}{|a|^2}\sup_{z\in\mathbb{C}} e^{p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)-p\varphi(z)} \int_{\mathbb{C}} |f(w)|^p e^{-p\varphi(w)}dA(w)\\
&= \frac{1}{|a|^2}\sup_{z\in\mathbb{C}} e^{p\varphi\left(|a||z|+\frac{2|b|}{|1-a|}\right)-p\varphi(|z|)}\|f\|^p_{(\varphi,p)}.
\end{aligned}$$

Hence

$$\|C_\psi^n\|_{(\varphi,p)} \le \frac{1}{|a|^2}\sup_{z\in\mathbb{C}} e^{\varphi\left(|a||z|+\frac{2|b|}{|1-a|}\right)-\varphi(|z|)}.$$

For the case $|a|=1$, if $f\in\mathcal{F}^p_\varphi$, then

$$\begin{aligned}
\|C_\psi^n f\|^p_{(\varphi,p)} &= \int_{\mathbb{C}} |f(\psi^n(z))|^p e^{-p\varphi(z)}dA(z)\\
&= \int_{\mathbb{C}} |f(a^n z)|^p e^{-p\varphi(a^n z)}dA(z) = \|f\|^p_{(\varphi,p)}.
\end{aligned}$$

From which we deduce that

$$\sup_{n\in\mathbb{N}}\|C_\psi^n\| = 1.$$

Case 2 : $p=\infty$.
For this we et $R_0=\frac{4|b|}{(1-|a|)|1-a|}$. Then for each $|z|\ge R_0$ and $n\in\mathbb{N}$,

$$|\psi^n(z)| = \left|a^n z+\frac{b(1-a^n)}{(1-a)}\right| \le |a||z|+\frac{2|b|}{|1-a|} \le \left(|a|+\frac{1-|a|}{2}\right)|z| < |z|. \tag{2.2}$$

Taking $f\in\mathcal{F}^\infty_\varphi$ with $\|f\|_{(\varphi,\infty)}=1$, we have $|f(z)|\le e^{\varphi(z)}$ for any $z\in\mathbb{C}$.
If $|a|<1$, applying (2.2),

$$\begin{aligned}
\sup_{|z|>R_0}|C_\psi^n f(z)|e^{-\varphi(z)} &= \sup_{|z|>R_0}\left|f\left(a^n z+\frac{b(1-a^n)}{1-a}\right)\right|e^{-\varphi(z)}\\
&\le \sup_{|z|>R_0} e^{\left(\varphi\left(a^n z+\frac{b(1-a^n)}{1-a}\right)-\varphi(z)\right)} \le 1,
\end{aligned}$$

and

$$\begin{aligned}
\sup_{|z|\le R_0}|C_\psi^n f(z)|e^{-\varphi(z)} &\le \sup_{|z|\le R_0}\left|f\left(a^n z+\frac{(b(1-a^n))}{1-a}\right)\right| \sup_{|z|\le R_0} e^{-\varphi(z)}\\
&= \max_{|z|=R_0}\left|f\left(a^n z+\frac{(b(1-a^n))}{1-a}\right)\right| \le \max_{|z|=R_0} e^{\varphi\left(a^n z+\frac{(b(1-a^n))}{1-a}\right)} \le e^{\varphi(R_0)}.
\end{aligned}$$

Thus for any $f \in \mathcal{F}_\varphi^p$

$$\|C_\psi^n f\|_{(\varphi,\infty)} < 1 + e^{\varphi(R_0)}.$$

From this we get that, for each $n \in \mathbb{N}$,

$$\|C_\psi^n\| = \sup_{\|f\|=1} \|C_\psi^n f\|_{(\varphi,\infty)} \leq 1 + e^{\varphi(R_0)}.$$

If $|a| = 1$, then

$$\|C_\psi^n f\|_{(\varphi,\infty)} = \sup_{z\in\mathbb{C}} |f(a^n z)|e^{-\varphi(z)} = \sup_{z\in\mathbb{C}} |f(a^n z)|e^{-\varphi(a^n z)} = \|f\|_{(\varphi,\infty)}, \tag{2.3}$$

which shows

$$\sup_{n\in\mathbb{N}} \|C_\psi^n\|_{(\varphi,\infty)} = 1.$$

□

Corollary 2.2. *Every bounded composition operator C_ψ on $\mathcal{F}_\varphi^0$ is power bounded.*

Proof. By Theorem 1.1, every bounded composition operator C_ψ on $\mathcal{F}_\varphi^\infty$ maps polynomials into polynomials. This implies by density that such operator C_ψ maps $\mathcal{F}_\varphi^0$ into itself. On the other hand, if C_ψ is bounded on $\mathcal{F}_\varphi^0$, then its bidual, which coincides with $C_\psi : \mathcal{F}_\varphi^\infty \to \mathcal{F}_\varphi^\infty$, is also bounded. Therefore, C_ψ is bounded on $\mathcal{F}_\varphi^0$ if and only if it is bounded on $\mathcal{F}_\varphi^\infty$. The conclusion follows from Theorem 2.1, since the restriction of a power bounded operator is clearly power bounded. □

Theorem 2.3. *The composition operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$, $\psi(z) = az + b,\ |a| < 1$, for $1 \leq p < \infty$ or $p = 0$, satisfies, for each $f \in \mathcal{F}_\varphi^p$,*

$$\lim_{n\to\infty} \|C_\psi^n f - C_{\frac{b}{1-a}} f\|_{(\varphi,p)} = 0.$$

Proof. Case 1: $1 \leq p < \infty$.
Since C_ψ is power bounded, there is a constant $M > 0$ such that

$$\|C_\psi^n f\|_{(\varphi,p)}^p \leq M\|f\|_{(\varphi,p)}^p.$$

Equivalently

$$\int_{\mathbb{C}} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right)\right|^p e^{-p\varphi(z)} dA(z) \leq \int_{\mathbb{C}} M^p |f(z)|^p e^{-p\varphi(z)} dA(z).$$

Moreover, by continuity of f,

$$\lim_{n\to\infty} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right) - f\left(\frac{b}{1-a}\right)\right| = 0. \tag{2.4}$$

Applying Dominated Convergent Theorem on the sequence

$$g_n(z) := \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right) - f\left(\frac{b}{1-a}\right)\right|^p e^{-p\varphi(z)},$$

we get that

$$\begin{aligned}\lim_{n\to\infty} \|C_\psi^n f - C_{\frac{b}{1-a}} f\|_{(\varphi,p)}^p \\ &= \lim_{n\to\infty} \int_{\mathbb{C}} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right) - f\left(\frac{b}{1-a}\right)\right|^p e^{-p\varphi(z)} dA(z) \\ &= \int_{\mathbb{C}} \lim_{n\to\infty} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right) - f\left(\frac{b}{1-a}\right)\right|^p e^{-p\varphi(z)} dA(z) = 0.\end{aligned}$$

Case 2: $p = 0$.

Fix $f \in \mathcal{F}_\varphi^0$. As in the proof of Theorem 2.1, there is $R_0 > 0$ such that, for each $|z| \geq R_0$ and each $n \in \mathbb{N}$, we have $\left|a^n z + \frac{b(1-a^n)}{1-a}\right| < |z|$. Hence

$$M_\infty(C_\psi^n f, r) e^{-\varphi(r)} \leq M_\infty(f, r) e^{-\varphi(r)}$$

for each $r \geq R_0$.

It is easy to see that

$$\psi^n(z) = a^n z + \frac{b(1-a^n)}{1-a} \to \frac{b}{1-a}$$

uniformly on compact subsets of $\mathbb{C}$. From this we have

$$f(\psi^n(z)) \to f\left(\frac{b}{1-a}\right)$$

uniformly on the compact subsets of $\mathbb{C}$. That is, for each compact set K in $\mathbb{C}$,

$$\sup_{z\in K} \left| f(\psi^n(z)) - f\left(\frac{b}{1-a}\right)\right| \to 0 \tag{2.5}$$

as $n \to \infty$.

Given $\varepsilon > 0$, since $f \in \mathcal{F}_\varphi^0$, we find $r_0 > R_0$ such that $M_\infty(f, r)e^{-\varphi(r)} < \varepsilon/2$ and $|f(\frac{b}{1-a})|e^{-\varphi(r)} < \varepsilon/2$ if $r > r_0$. Then, for each $|z| > r_0$ and $n \in \mathbb{N}$, we have

$$\left| C_\psi^n f(z) - f\left(\frac{b}{1-a}\right)\right| e^{-\varphi(z)} \leq M_\infty(f, |z|) e^{-\varphi(z)} + \left| f\left(\frac{b}{1-a}\right)\right| e^{-\varphi(z)} < \frac{\varepsilon}{2}.$$

We apply (2.5) to the compact set $K_0 = \{z \in \mathbb{C} \mid |z| \leq r_0\}$ to find n_0 such that if $z \in K_0$ and $n \geq n_0$ we have

$$\left| f(\psi^n(z)) - f\left(\frac{b}{1-a}\right)\right| < \frac{\varepsilon}{2S},$$

with $S := \max_{z\in K_0} e^{-\varphi(z)}$. If $n \geq n_0$ and $z \in \mathbb{C}$, we have

$$\left| C_\psi^n f(z) - f(\frac{b}{1-a})\right| e^{-\varphi(z)} < \varepsilon.$$

Thus

$$\lim_{n\to\infty} \|C_\psi^n f - C_{\frac{b}{1-a}} f\|_{(\varphi,\infty)} = 0.$$

□

Corollary 2.4. *The composition operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$, $\psi(z) = az + b$, $|a| < 1$, for $1 \leq p < \infty$ or $p = 0$, is mean ergodic.*

Proof. Theorem 2.3 implies, for each $f \in \mathcal{F}_\varphi^p$,

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=1}^{n} C_\psi^k f - C_{\frac{b}{1-a}} f \right\|_{(\varphi,p)} = 0,$$

from where the conclusion follows. □

We observe that if $1 < p < \infty$, then the space $\mathcal{F}_\varphi^p$ is a reflexive Banach space. Since every bounded operator C_ψ is power bounded by Theorem 2.1, it follows directly from Yosida's mean ergodic theorem (Theorem VIII.3.2 in [22]) that every bounded operator C_ψ is mean ergodic on $\mathcal{F}_\varphi^p, 1 < p < \infty$.

An abstract result of Yosida and Kakutani ([23] Theorem 4 and Corollary on page 204-205 and Theorem 2.8 in [13]) implies that every compact power bounded operator on a Banach space is uniformly mean ergodic. This fact permits us to improve Corollary 2.4.

Theorem 2.5. *Let $\psi(z) = az + b, |a| < 1$. Then the operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$, $1 \le p \le \infty$ or $p = 0$, is uniformly mean ergodic and*

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=1}^{n} C_\psi^k - C_{\frac{b}{1-a}} \right\| = 0. \tag{2.6}$$

Proof. The operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$, $1 \le p \le \infty$ or $p = 0$, defined by $\psi(z) = az + b, |a| < 1$ is compact by Theorem 1.1, and it is power bounded by Theorem 2.1. Therefore, C_ψ is uniformly mean ergodic for $1 \le p \le \infty$ by the Theorem of Yosida and Kakutani ([23] Theorem 4 and Corollary on page 204-205 and Theorem 2.8 in [13]). Combining this with Corollary 2.4 we get, for $1 \le p < \infty$ or $p = 0$,

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=1}^{n} C_\psi^k - C_{\frac{b}{1-a}} \right\| = 0.$$

Next, we show (2.6) for $p = \infty$. Since $\mathcal{F}_\varphi^\infty$ is canonically isomorphic to the bidual of $\mathcal{F}_\varphi^0$, and the bitranspose operator $C_\psi^{''}$ of $C_\psi : \mathcal{F}_\varphi^0 \to \mathcal{F}_\varphi^0$ coincides with composition operator $C_\psi : \mathcal{F}_\varphi^\infty \to \mathcal{F}_\varphi^\infty$, the conclusion follows from the well-known fact that $\|T\| = \|T^{'}\| = \|T^{''}\|$ for any bounded operator T on a Banach space. □

It is well-known that every periodic operator is uniformly mean ergodic. We include the proof of the next result for the sake of completeness.

Proposition 2.6. *The composition operator C_ψ, where $\psi(z) = e^{i\theta}z$, $0 < \theta \le 2\pi$ and $\frac{2k\pi}{\theta} = m$ for some positive integers m, k, is uniformly mean ergodic on $\mathcal{F}_\varphi^p$, $1 \le p \le \infty$ or $p = 0$. Moreover, in this case every $f \in \mathcal{F}_\varphi^p$ is a periodic point of C_ψ.*

Proof. In this case the sequence C_ψ^n is periodic with period m. Any $n \in \mathbb{N}$ can be written in the form of $n = ml + j$ for some $l \in \mathbb{N}$ and $j = 0, 1, 2, ..., m-1$. Thus

$$\begin{aligned}\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}C_\psi^k - \frac{1}{m}\sum_{k=1}^{m}C_\psi^k\right\|_{(\varphi,p)} &= \lim_{l\to\infty}\frac{1}{(ml+j)}\left\|\sum_{k=1}^{j}C_\psi^k - \frac{j}{m}\sum_{k=1}^{m}C_\psi^k\right\|_{(\varphi,p)} \\ &\le \lim_{l\to\infty}\frac{1}{(ml+j)}\left(\sum_{k=1}^{j}\|C_\psi^k\|_{(\varphi,p)} + \frac{j}{m}\sum_{k=1}^{m}\|C_\psi^k\|_{(\varphi,p)}\right) = 0.\end{aligned}$$

The statement about the periodic points of C_ψ is trivial. □

It remains to treat for the case when $\frac{2\pi}{\theta}$ is not rational number.

Theorem 2.7. *If $\varphi(z) = az$, $|a| = 1$, and a is not root of unity, then the composition operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$, $1 \le p < \infty$ or $p = 0$ is mean ergodic but not uniformly mean ergodic.*

Proof. First we show that

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}C_\psi^k f - C_0 f\right\|_{(\varphi,p)} = 0 \tag{2.7}$$

when f is a polynomial. It is enough to check it when f belongs to the sequence of monomials $\{1, z, z^2, ...\}$. If $f(z) = 1$, then $C_\varphi f$ is the constant function 1, and hence (2.7) holds. If $f(z) = z^m$ for some $m \in \mathbb{N}$,

$$\begin{aligned}\left\|\frac{1}{n}\sum_{k=1}^{n}C_\psi^k f\right\|_{(\varphi,p)} &= \left\|\frac{1}{n}\sum_{k=1}^{n}a^{mk}z^m\right\|_{(\varphi,p)} \le \left(\frac{1}{n}\sum_{k=1}^{n}|a|^{mk}\right)\|z^m\|_{(\varphi,p)} \\ &= \frac{|a^m||1-a^{mn}|}{n|1-a^m|}\|z^m\|_{(\varphi,p)} \le \frac{2}{n|1-a^m|}\|z^m\|_{(\varphi,p)} \to 0\end{aligned}$$

as $n \to \infty$. Since the set of polynomials is dense in $\mathcal{F}_\varphi^p$ and C_φ is power bounded (and hence $\{C_\varphi^n\}_n$ is equicontinuous) on $\mathcal{F}_\varphi^p$ by Theorem 2.1, we have (see e.g. Lemma 2.1 in [2]),

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}C_\psi^k f - C_0 f\right\|_{(\varphi,p)} = 0$$

for every $f \in \mathcal{F}_\varphi^p$. This implies that $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$ is mean ergodic.

Next, we prove that C_ψ is not uniformly mean ergodic. By Theorem 2.6 in [18], $\sigma(C_\psi) = \overline{\{a^n : n = 0, 1, 2, ...\}}$. This shows $1 \in \sigma(C_\psi)$ and 1 is an accumulation point of $\sigma(C_\psi)$. Therefore by Theorem 2.7 in [13] (Theorem 3.16 in [11]), C_ψ is not uniformly mean ergodic. □

Recall that a Banach space X is a *Grothendieck space* if every sequence (x_n) in X' which is convergent to 0 for the weak topology $\sigma(X', X)$ is also convergent to 0 for the weak topology $\sigma(X', X'')$. The space X has the *Dunford-Pettis property* if for any sequence (x_n) in X which is convergent to 0 for the weak topology $\sigma(X, X')$ and any sequence (x_n') in X' which is convergent to 0 for the weak topology one gets $\sigma(X', X'')$ one gets $\lim_{n\to\infty}\langle x_n, x_n'\rangle$. The spaces ℓ^∞ or $H^\infty(\mathbb{D})$ are examples of Grothendieck spaces with Dunford-Pettis property [14].

Theorem 2.8. *If $\varphi(z) = az$, $|a| = 1$, and a is not root of unity, the composition operator $C_\psi : \mathcal{F}_\varphi^\infty \to \mathcal{F}_\varphi^\infty$, is not mean ergodic, hence not uniformly mean ergodic.*

Proof. Since $1 \in \sigma(C_\psi) = \overline{\{a^n : n = 0, 1, 2, ...\}}$ and 1 is an accumulation point of $\sigma(C_\psi)$, we apply Theorem 2.7 in [13] (Theorem 3.16 in [11]) to conclude that C_ψ is not uniformly mean ergodic on $\mathcal{F}_\varphi^\infty$.

On the other hand, Theorem 1.1 in [16] implies that $\mathcal{F}_\varphi^\infty$ is isomorphic to ℓ^∞ or $H^\infty(\mathbb{D})$. Hence $\mathcal{F}_\varphi^\infty$ is Grothendieck spaces with Dunford-Pettis property. By a result of Lotz [15] every power bounded mean ergodic operator on a Grothendieck Banach space with the Dunford-Pettis property is uniformly mean ergodic. Therefore that C_ψ is not mean ergodic in $\mathcal{F}_\varphi^\infty$. □

We say that a point x in X is periodic under an operator T on X if there is some $n \in \mathbb{N}$ such that $T^n x = x$.

Lemma 2.9. *Let $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, $\alpha^n \neq 1$ for every $n \in \mathbb{N}$ and f is analytic function on $\mathbb{C}$. If $f(\alpha z + \beta) = f(z)$ for each $z \in \mathbb{C}$, then f is constant.*

Proof. The mapping $\phi(z) = \alpha z + \beta$ fixes the point $z_0 := \frac{\beta}{1-\alpha}$. Define $g(z) = f(z) - f(z_0)$. Then $g(z_0) = 0$ and $g^{(n)}(z) = f^{(n)}(z)$ for every $n \in \mathbb{N}$.
Since $f(\alpha z+\beta) = f(z)$ for each $z \in \mathbb{C}$, we have $\alpha^n f^{(n)}(\alpha z+\beta) = f^{(n)}(z)$, and hence $\alpha^n f^{(n)}(z_0) = f^{(n)}(z_0)$. This yields $f^{(n)}(z_0) = 0$ for every $n \in \mathbb{N}$ since $\alpha^n \neq 1$. Thus $g^{(n)}(z_0) = 0$ for every $n = 0, 1, 2, ...$ Hence $g(z) = 0$ for every $z \in \mathbb{C}$. Therefore, $f(z) = f(z_0)$ for every $z \in \mathbb{C}$. □

Observe that the assumption that $\alpha^n \neq 1$ for every $n \in \mathbb{N}$ is necessary in Lemma 2.9. For example if we take $\alpha = 1$ and $\beta = i2\pi$, then the function $f(z) = e^z$ satisfies $f(\alpha z + \beta) = f(z + i2\pi) = e^{z+i2\pi} = e^z = f(z)$ for each $z \in \mathbb{C}$.

Proposition 2.10. *Let C_ψ be a bounded composition operator on $\mathcal{F}_\varphi^p$, $1 \leq p \leq \infty$ or $p = 0$.*

(a) *Every $f \in \mathcal{F}_\varphi^p$ is periodic point for C_ψ if $\psi(z) = az, |a| = 1$ and $a^m = 1$ for some $m \in \mathbb{N}$.*
(b) *Only constant functions in $\mathcal{F}_\varphi^p$ are periodic points for C_ψ if $\psi(z) = az + b, |a| < 1$ or $\psi(z) = az, |a| = 1, a^n \neq 1$ for all $n \in \mathbb{N}$.*

Proof. (a) If $\psi(z) = az, |a| = 1$ and $a^m = 1$ for some $m \in \mathbb{N}$, then for each $f \in \mathcal{F}_\varphi^p$, $C_\psi^m f(z) = f(a^m z) = f(z)$ for every $z \in \mathbb{C}$.
(b) Assume that $\psi(z) = az + b, |a| < 1$ and $b \in \mathbb{C}$ or $\psi(z) = az, |a| = 1, a^n \neq 1$ for all $n \in \mathbb{N}$. If $f \in \mathcal{F}_\varphi^p$ is a periodic point of C_ψ, there is $s \in \mathbb{N}$ such that $C_\psi^s f = f$. Then

$$C_\psi^s f(z) = f(\psi^s(z)) = f\left(a^s z + \frac{b(1 - a^s)}{1 - a}\right) = f(z).$$

Our assumptions on the symbol ψ imply that $(a^s)^n \neq 1$ for each $n \in \mathbb{N}$. We can apply Lemma 2.9 to conclude that f must be constant. □

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